

X. *On the Motion of Bodies affected by Friction.* By the Rev. Samuel Vince, A. M. of Cambridge; communicated by Anthony Shepherd, D. D. F. R. S. Plumian Professor of Astronomy and experimental Philosophy at Cambridge.

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THE subject of the paper which I have now the honour of presenting to the Royal Society, seems to be of a very considerable importance both to the practical mechanic and to the speculative philosopher; to the former, as a knowledge of the laws and quantity of the friction of bodies in motion upon each other will enable him at first to render his machines more perfect, and save him in a great measure the trouble of correcting them by trials; and to the latter, as those laws will furnish him with principles for his theory, which when established by experiments will render his conclusions applicable to the real motion of bodies upon each other. But, however important a part of mechanics this subject may constitute, and however, from its obvious uses, it might have been expected to have claimed a very considerable attention both from the mechanic and philosopher, yet it has, of all the other parts of this branch of natural philosophy, been the most neglected. The law by which the motions of bodies are retarded by friction has never, that I know of, been truly established. MUSSCHENBROEK says, that in small velocities the friction varies very nearly as the velocity, but that in great velocities the friction increases; he has also attempted to prove, that by increasing the

the weight of a body the friction does not always increase exactly in the same ratio; and that the same body, if by changing its position you change the magnitude of the surface on which it moves, will have its quantity of friction also changed. HELSHAM and FERGUSON, from the same kind of experiments, have endeavoured to prove, that the friction does *not* vary by changing the quantity of surface on which the body moves; and the latter of these asserts, that the friction increases very nearly as the velocity; and that by increasing the weight, the friction is increased in the *same* ratio. These different conclusions induced me to repeat their experiments, in order to see how far they were conclusive in respect to the principles deduced from them: when it appeared, that there was another cause operating besides friction, which they had not attended to, and which rendered all their deductions totally inconclusive. Of those who have written on the theory, no one has established it altogether on true principles: EULER (whose theory is extremely elegant, and which, as he has so fully considered the subject, would have precluded the necessity of offering any thing further, had its principles been founded on experiments) supposes the friction to vary in proportion to the velocity of the body, and its pressure upon the plane, neither of which are true: and others, who have imagined that friction is a uniformly retarding force (and which conjecture will be confirmed by our experiments), have still retained the other supposition, and therefore rendered their solutions not at all applicable to the cases for which they were intended. I therefore endeavoured by a set of experiments to determine,

1st, *Whether friction be a uniformly retarding force,*

2dly, *The quantity of friction.*

3dly,

3dly, *Whether the friction varies in proportion to the pressure or weight.*

4thly, *Whether the friction be the same on whichever of its surfaces a body moves.*

The experiments, in which I was assisted by my ingenious friend the Rev. Mr. JONES, Fellow of Trinity College, were made with the utmost care and attention, and the several results agreed so very exactly with each other, that I do not scruple to pronounce them to be conclusive.

2. A plane was adjusted parallel to the horizon, at the extremity of which was placed a pulley, which could be elevated or depressed in order to render the string which connected the body and the moving force parallel to the plane. A scale accurately divided was placed by the side of the pulley perpendicular to the horizon, by the side of which the moving force descended; upon the scale was placed a moveable stage, which could be adjusted to the space through which the moving force descended in any given time, which time was measured by a well regulated pendulum clock vibrating seconds. Every thing being thus prepared, the following experiments were made to ascertain the law of friction. But let me first observe, that if friction be a uniform force, the difference between it and the given force of the moving power must be also uniform, and therefore the moving body must descend with a uniformly accelerated velocity, and consequently the spaces described from the beginning of the motion must be as the squares of the times, just as when there was no friction, only they will be diminished on account of the friction.

3. EXP. 1. A body was placed upon the horizontal plane, and a moving force applied, which from repeated trials was found to descend $52\frac{1}{2}$ inches in 4'', for by the beat of the clock and

the found of the moving force when it arrived at the stage, the space could be very accurately adjusted to the time; the stage was then removed to that point to which the moving force would descend in 3'', upon supposition that the spaces described by the moving power were as the squares of the times; and the space was found to agree very accurately with the time; the stage was then removed to that point to which the moving force ought to descend in 2'', upon the same supposition, and the descent was found to agree exactly with the time; lastly, the stage was adjusted to that point to which the moving force ought to descend in 1'', upon the same supposition, and the space was observed to agree with the time. Now, in order to find whether a difference in the time of descent could be observed, by removing the stage a little above and below the positions which corresponded to the above times, the experiment was tried, and the descent was always found too soon in the former, and too late in the latter case; by which I was assured that the spaces first mentioned corresponded exactly to the times. And, for the greater certainty, each descent was repeated eight or ten times; and every caution used in this experiment was also made use of in all the following.

EXP. 2. A second body was laid upon the horizontal plane, and a moving force applied which descended $41\frac{1}{4}$ inches in 3''; the stage was then adjusted to the space corresponding to 2'', upon supposition that the spaces descended through were as the squares of the times, and it was found to agree accurately with the time; the stage was then adjusted to the space corresponding to 1'', upon the same supposition, and it was found to agree with the time.

EXP. 3. A third body was laid upon the horizontal plane, and a moving force applied, which descended $59\frac{1}{4}$ inches in 4''; the
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stage was then adjusted to the space corresponding to $3''$, upon supposition that the spaces descended through were as the squares of the times, and it was found to agree with the time; the stage was then adjusted to the space corresponding to $2''$, upon the same supposition, and it was found to agree with the time; the stage was then adjusted to the space corresponding to $1''$, and was found to agree with the time.

EXP. 4. A fourth body was then taken and laid upon the horizontal plane, and a moving force applied, which descended 55 inches in $4''$; the stage was then adjusted to the space through which it ought to descend in $3''$, upon supposition that the spaces descended through were as the squares of the times, and it was found to agree with the time; the stage was then adjusted to the space corresponding to $2''$, upon the same supposition, and was found to agree with the time; lastly, the stage was adjusted to the space corresponding to $1''$, and it was found to agree exactly with the time.

Besides these experiments, a great number of others were made with hard bodies, or those whose parts so firmly cohered as not to be moved *inter se* by the friction; and in each experiment bodies of very different degrees of friction were chosen, and the results all agreed with those related above; we may therefore conclude, that *the friction of hard bodies in motion is a uniformly retarding force.*

But to determine whether the same was true for bodies when covered with cloth, woollen, &c. experiments were made in order to ascertain it; when it was found in all cases, that the retarding force increased with the velocity; but, upon covering bodies with paper, the consequences were found to agree with those related above.

4. Having proved that the retarding force of all hard bodies arising from friction is uniform, the quantity of friction, considered as equivalent to a weight without inertia drawing the body on the horizontal plane backwards, or acting contrary to the moving force, may be immediately deduced from the foregoing experiments. For let M = the moving force expressed by its weight; F = the friction; W = the weight of the body upon the horizontal plane; S = the space through which the moving force descended in the time t expressed in seconds; $r = 16\frac{1}{2}$ feet; then the whole accelerative force (the force of gravity being unity) will be $\frac{M-F}{M+W}$; hence, by the laws of uniformly accelerated motions, $\frac{M-F}{M+W} \times r t^2 = S$, consequently $F = M - \frac{M+W \times S}{r t^2}$. To exemplify this, let us take the case of the last experiment, where $M = 7$, $W = 25\frac{3}{4}$, $S = 4\frac{7}{2}$ feet, $t = 4''$; hence $F = 7 - \frac{32\frac{3}{4} \times 4\frac{7}{2}}{16\frac{1}{2} \times 16} = 6.417$; consequently the friction was to the weight of the rubbing body as 6.4167 to 25.75. And the great accuracy of determining the friction by this method is manifest from hence, that if an error of 1 inch had been made in the descent (and experiments carefully made may always determine the space to a much greater exactness) it would not have affected the conclusion $\frac{1}{6000}$ th part of the whole.

5. We come in the next place to determine, whether friction, *cæteris paribus*, varies in proportion to the weight or pressure. Now if the whole quantity of the friction of a body, measured by a weight without inertia equivalent to the friction drawing the body backwards, increases in proportion to its weight, it is manifest, that the retardation of the velocity of the body arising from the friction will not be altered; for the

retardation varies as $\frac{\text{Quantity of friction}}{\text{Quantity of matter}}$; hence, if a body be put in motion upon the horizontal plane by any moving force, if both the weight of the body and the moving force be increased in the same ratio, the acceleration arising from that moving force will remain the same, because the accelerative force varies as the moving force divided by the whole quantity of matter, and both are increased in the same ratio; and if the quantity of friction increases also as the weight, then the retardation arising from the friction will, from what has been said, remain the same, and therefore the whole acceleration of the body will not be altered; consequently the body ought, upon this supposition, still to describe the same space in the same time. Hence, by observing the spaces described in the same time, when both the body and the moving force are increased in the same ratio, we may determine whether the friction increases in proportion to the weight. The following experiments were therefore made in order to ascertain this matter.

EXP. 1. A body weighing 10 oz. by a moving force of 4 oz. described in 2'' a space of 51 inches; by loading the body with 10 oz. and the moving force with 4 oz. it described 56 inches in 2''; and by loading the body again with 10 oz. and the moving force with 4 oz. it described 63 inches in 2''.

EXP. 2. A body, whose weight was 16 oz. by a moving force of 5 oz. described a space of 49 inches in 3''; and by loading the body with 64 oz. and the moving force with 20 oz. the space described in the same time was 64 inches.

EXP. 3. A body weighing 6 oz. by a moving force of $2\frac{1}{2}$ oz. described 28 inches in 2''; and by loading the body with 24 oz. and the moving force with 10 oz. the space described in the same time was 54 inches.

EXP. 4. A body weighing 8 oz. by a moving force of 4 oz. described $33\frac{1}{2}$ inches in 2''; and by loading the body with 8 oz. and the moving force with 4 oz. the space described in the same time was 47 inches.

EXP. 5. A body whose weight was 9 oz. by a moving force of $4\frac{1}{2}$ oz. described 48 inches in 2''; and by loading the body with 9 oz. and the moving force with $4\frac{1}{2}$ oz. the space described in the same time was 60 inches.

EXP. 6. A body weighing 10 oz. by a moving force of 3 oz. described 20 inches in 2''; by loading the body with 10 oz. and the moving force with 3 oz. the space described in the same time was 31 inches; and by loading the body again with 30 oz. and the moving force with 9 oz. the space described was 34 inches in 2''.

From these experiments, and many others which it is not necessary here to relate, it appears, that the space described is always increased by increasing the weight of the body and the accelerative force in the same ratio; and as the acceleration arising from the moving force continued the same, it is manifest, that the retardation arising from the friction must have been diminished, for the whole accelerative force must have been increased on account of the increase of the space described in the same time; and hence (as the retardation from friction varies as $\frac{\text{Quantity of friction}}{\text{Quantity of matter}}$) *the quantity of friction increases in a less ratio than the quantity of matter or weight of the body.*

6. We come now to the last thing which it was proposed to determine, that is, whether the friction varies by varying the surface on which the body moves. Let us call two of the surfaces A and a, the former being the greater, and the latter the less. Now the weight on every given part of a is as much greater
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than the weight on an equal part of A , as A is greater than a ; if therefore the friction was in proportion to the weight, *cæteris paribus*, it is manifest, that the friction on a would be equal to the friction on A , the whole friction being, upon such a supposition, as the weight on any given part of each surface multiplied into the number of such parts or into the whole area, which products, from the proportion above, are equal. But from the last experiments it has been proved, that the friction on any given surface increases in a less ratio than the weight; consequently the friction on any given part of a has a less ratio to the friction on an equal part of A than A has to a , and hence the friction on a is less than the friction on A , that is, the smallest surface has always the least friction. But as this conclusion is contrary to the generally received opinion, I have thought it proper to confirm the same by a set of experiments. But before I proceed to relate them, I will beg leave to recommend to those, who may afterwards be induced to repeat them, the following cautions, which are extremely necessary to be attended to. Great care must be taken that the two surfaces have exactly the same degree of roughness; in order to be certain of which, such bodies must be chosen as have no knots in them, and whose grain is so very regular that when the two surfaces are planed with a fine rough plane, their roughness may be the same, which will not be the case if the body be knotty, or the grain irregular, or if it happens not to run in the same direction on both surfaces. When you cannot depend on the surfaces having the same degree of roughness, the best way will be to paste some fine rough paper on each surface, which perhaps will give a more equal degree of roughness than can be obtained by any other method. Now as the proof which I have already given depends only on the motion
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of the body upon the *same* surface, it is not liable to any inaccuracy of the kind which the preceding cautions have been given to avoid, nor indeed to any other, and therefore it must be perfectly conclusive. In the following experiments the cautions mentioned above were carefully attended to.

EXP. 1. A body was taken whose flat surface was to its edge as 22 : 9, and with the same moving force the body described on its flat side $33\frac{1}{2}$ inches in 2'', and on its edge 47 inches in the same time.

EXP. 2. A second body was taken whose flat surface was to its edge as 32 : 3, and with the same moving force it described on its flat side 32 inches in 2'', and on its edge it described $37\frac{1}{2}$ inches in the same time.

EXP. 3. I took another body and covered one of its surfaces, whose length was 9 inches, with a fine rough paper, and by applying a moving force, it described 25 inches in 2''; I then took off some paper from the middle, leaving only $\frac{3}{4}$ of an inch at the two ends, and with the same moving force it described 40 inches in the same time.

EXP. 4. Another body was taken which had one of its surfaces, whose length was 9 inches, covered with a fine rough paper, and by applying a moving force it described 42 inches in 2''; some of the paper was then taken off from the middle, leaving only $1\frac{3}{8}$ inches at the two ends, and with the same moving force it described 54 inches in 2''; I then took off more paper, leaving only $\frac{1}{4}$ of an inch at the two ends, and the body then described, by the same moving force, 60 inches in the same time.

In the two last experiments the paper which was taken off the surface was laid on the body, that its weight might not be altered.

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EXP. 5. A body was taken whose flat surface was to its edge as 30 : 17; the *flat* side was laid upon the horizontal plane, a moving force was applied, and the stage was fixed in order to stop the moving force, in consequence of which the body would then go on with the velocity acquired until the friction had destroyed all its motion; when it appeared from a mean of 12 trials that the body moved, after its acceleration ceased, $5\frac{1}{4}$ inches before it stopped. The *edge* was then applied, and the moving force descended through the same space, and it was found, from a mean of the same number of trials, that the space described was $7\frac{1}{2}$ inches before the body lost all its motion, after it ceased to be accelerated.

EXP. 6. Another body was then taken whose flat surface was to its edge as 60 : 19, and, by proceeding as before, on the flat surface it described, at a mean of 12 trials, $5\frac{1}{8}$ inches, and on the edge $6\frac{1}{2}\frac{1}{4}$ inches, before it stopped, after the acceleration ceased.

EXP. 7. Another body was taken whose flat surface was to its edge as 26 : 3, and the spaces described on these two surfaces, after the acceleration ended, were, at a mean of 10 trials, $4\frac{3}{7}$ and $7\frac{7}{8}$ inches respectively.

From all these different experiments it appears, that the smallest surface had always the least friction, which agrees with the consequence deduced from the consideration that the friction does not increase in so great a ratio as the weight; we may therefore conclude, that *the friction of a body does not continue the same when it has different surfaces applied to the plane on which it moves, but that the smallest surface will have the least friction.*

7. Having thus established, from the most decisive experiments, all that I proposed relative to friction, I think it proper, before

before I conclude, to give the result of my examination into the nature of the experiments which have been made by others; which were repeated, in order to see how far they were conclusive in respect to the principles which have been deduced from them. The experiments which have been made by all the authors that I have seen, have been thus instituted. To find what moving force would *just* put a body at rest in motion: and they concluded from thence, that the accelerative force was then equal to the friction; but it is manifest, that any force which will put a body in motion must be *greater* than the force which opposes its motion, otherwise it could not overcome it; and hence, if there were no other objection than this, it is evident, that the friction could not be very accurately obtained; but there is another objection which totally destroys the experiment so far as it tends to show the quantity of friction, which is the strong cohesion of the body to the plane when it lies at rest; and this is confirmed by the following experiments. 1st, A body of $12\frac{1}{4}$ oz. was laid upon an horizontal plane, and then loaded with a weight of 8 lb. and such a moving force was applied as would, when the body was *just put* in motion, continue that motion without any acceleration, in which case the friction must be just equal to the accelerative force. The body was then stopped, when it appeared, that the same moving force which had *kept* the body in motion before, would not *put* it in motion, and it was found necessary to take off $4\frac{1}{2}$ oz. from the body before the same moving force *would* put it in motion; it appears, therefore, that this body, when laid upon the plane at rest, acquired a very strong cohesion to it. 2dly, A body whose weight was 16 oz. was laid at rest upon the horizontal plane, and it was found that a moving force of 6 oz. would *just put* it in motion; but that a moving force of 4 oz. *would*

would, when it was just put in motion, *continue* that motion without any acceleration, and therefore the accelerative force must *then* have been equal to the friction, and not when the moving force of 6 oz. was applied.

From these experiments therefore it appears, how very considerable the cohesion was in proportion to the friction when the body was in motion; it being, in the latter case, almost $\frac{1}{3}$, and in the former it was found to be very nearly equal to the whole friction. All the conclusions therefore deduced from the experiments, which have been instituted to determine the friction from the force necessary to *put* a body in motion (and I have never seen any described but upon such a principle) have manifestly been totally false; as such experiments only shew the resistance which arises from the cohesion and friction conjointly.

8. I shall conclude this part of the subject with a remark upon Art. 5. It appears from all the experiments which I have made, that the proportion of the increase of the friction to the increase of the weight was different in all the different bodies which were made use of; no general rule therefore can be established to determine this for *all* bodies, and the experiments which I have hitherto made have not been sufficient to determine it for the *same* body. At some future opportunity, when I have more leisure, I intend to repeat the experiments in order to establish, in some particular cases, the law by which the quantity of friction increases by increasing the weight. Leaving this subject therefore for the present, I shall proceed to establish a theory upon the principles which we have already deduced from our experiments.

PROPOSITION I.

Let e, f, g, (fig. 1.) represent either a cylinder, or that circular section of a body on which it rolls down the inclined plane CA in consequence of its friction, to find the time of descent and the number of revolutions.

As it has been proved in Art. 5. that the friction of a body does not increase in proportion to its weight or pressure, we cannot therefore, by knowing the friction on any other plane, determine the friction on CA; the friction therefore on CA can only be determined by experiments made upon *that* plane, that is, by letting the body descend from rest, and observing the space described in the first second of time; call that space a , and then, as by Art. 3. friction is a uniformly retarding force, the body must be uniformly accelerated, and consequently the whole time of descent in seconds will be $= \sqrt{\frac{AC}{a}}$. Now to determine the number of revolutions, let s be the center of oscillation to the point of suspension a^* ; then, because no force acting at a can affect the motion of the point s , that point, notwithstanding the action of the friction at a , will always have a motion parallel to CA uniformly accelerated by a force equal to that with which the body would be accelerated if it had no friction; hence, if $2m = 32\frac{1}{2}$ feet, the velocity acquired by the point s in the first second will be $= \frac{2m \times CB}{CA}$; now the excess of the ve-

* a and s are not fixed points in the body, but the former always represents that point of the body in contact with the plane, and the latter the corresponding center of oscillation.

velocity of the point s above that of r (r being the center) is manifestly the velocity with which s is carried about r ; hence the velocity of s about the center $= \frac{2m \times CB}{CA} - 2a = \frac{2m \times CB - 2a \times CA}{CA}$, consequently $rs : ra :: \frac{2m \times CB - 2a \times CA}{CA} : \frac{2m \times ra \times CB - 2a \times ra \times CA}{rs \times CA}$ $=$ the velocity with which a point of the circumference is carried about the center, and which therefore expresses the force which accelerates the rotation; now as $2a$ expresses the accelerative force of the body down the plane, and the spaces described in the same time are in proportion to those forces, we have $2a : CA :: \frac{2m \times ra \times CB - 2a \times ra \times CA}{rs \times CA} : \frac{m \times ra \times CB - a \times ra \times CA}{a \times rs}$ the space which any point of the circumference describes about the center in the whole time of the body's descent down CA ; which being divided by the circumference $p \times ra$ (where $p=6.283$ &c.) will give $\frac{m \times BC - a \times AC}{p \times a \times rs}$ for the whole number of revolutions required.

Cor. 1. If $a \times CA = m \times BC$, the number of revolutions $= 0$, and therefore the body will then only slide; consequently the friction vanishes.

Cor. 2. Let $a'r's'$ (fig. 2.) be the next position of ars , and draw $tr'b$ parallel to sa , then will $s't$ represent the retardation of the center r arising from friction, and $a'b$ will represent the acceleration of a point of the circumference about its center; hence the retardation of the center : acceleration of the circumference about the center $:: s't : a'b ::$ (by sim. Δ 's) $tr' : br' :: rs : ra$.

Cor. 3. If a' coincides with a , the body does not *slide* but only *roll*; now in this case $ss' : rr' :: as : ar$; but as ss' and rr' represent the ratio of the velocities of the points s and r ,

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they will be to each other as $\frac{2m \times BC}{CA} : 2a$ or as $m \times CB : a \times CA$; hence, when the body *rolls* without *sliding*, $as : ar :: m \times CB : a \times CA$.

Cor. 4. The time of descent down CA is $= \sqrt{\frac{AC}{a}}$; but by the last Cor. when the body *rolls* without *sliding*, $a = \frac{m \times ra \times BC}{sa \times AC}$, hence the time of descent in that case $= AC \sqrt{\frac{sa}{m \times ra \times BC}}$; now the time of descent, if there were no friction, would be $= \frac{AC}{\sqrt{m \times BC}}$, hence the time of descent, when the body *rolls* without *sliding* : time of free descent $:: \sqrt{sa} : \sqrt{ra}$.

Cor. 5. By the last Cor. it appears, that when the body just *rolls* without *sliding*, or when the friction is just equal to the accelerative force, the time of descent $= AC \sqrt{\frac{sa}{m \times ra \times BC}}$; now it is manifest, that the time of descent will continue the same, if the friction be increased, for the body will still freely roll, as no increase of the friction acting at a can affect the motion of the point s .

If the body be projected from C with a velocity, and at the same time have a rotatory motion, the time of descent and the number of revolutions may be determined from the common principles of uniformly accelerated motions, as we have already investigated the accelerative force of the body down the plane and of its rotation about its axis; it seems therefore unnecessary to lengthen out this paper with the investigations.

PROPOSITION II.

Let the body be projected on an horizontal plane LM (fig. 3.) with a given velocity, to determine the space through which the body will move before it stops, or before its motion becomes uniform.

CASE I. 1. Suppose the body to have no rotatory motion when it begins to move; and let a = the velocity of projection per second measured in feet, and let the retarding force of the friction of the body, measured by the velocity of the body which it can destroy in one second of time, be determined by experiment and called F , and let x be the space through which the body would move by the time its motion was all destroyed when projected with the velocity a , and retarded by a force F ; then, from the principles of uniformly retarded motion, $x = \frac{a^2}{2F}$, and if t = time of describing that space, we have $t = \frac{a}{F}$, and hence the space described in the first second of time $= \frac{2a - F}{2}$. Now it is manifest, that when the rotatory motion of the body about its axis is equal to its progressive motion, the point a will be carried backwards by the *former* motion as much as it is carried forwards by the *latter*; consequently the point of contact of the body with the plane will then have no motion in the direction of the plane, and hence the friction will at that instant cease, and the body will continue to *roll* on uniformly without *sliding* with the velocity which it has at that point. Put therefore x = the space described from the commencement of the motion till it becomes uniform, then the body being uniformly retarded, the spaces from the end of the

the motion vary as the squares of the velocities, hence $\frac{a^2}{2F} : a^2 (:: 1 : 2F) :: \frac{a^2}{2F} - z : a^2 - 2Fz = \text{square of the progressive velocity when the motion becomes uniform; therefore the velocity destroyed by friction} = a - \sqrt{a^2 - 2Fz}$; hence, as the velocity generated or destroyed in the same time is in proportion to the force, we have by Cor. 2. Prop. 1. $rs : ra :: a - \sqrt{a^2 - 2Fz} : \frac{ra}{rs} \times a - \sqrt{a^2 - 2Fz}$ the velocity of the circumference efg generated about the center, consequently $\sqrt{a^2 - 2Fz} = \frac{ra}{rs} \times a - \sqrt{a^2 - 2Fz}$, and hence $z = \frac{rs^2 + 2rs \times ra \times a^2}{as^2 \times 2F}$ the space which the body describes before the motion becomes uniform.

2. If we substitute this value of z into the expression for the velocity, we shall have $a \times \frac{ra}{rs}$ for the velocity of the body when its motion becomes uniform; hence therefore it appears, that the velocity of the body, when the friction ceases, will be the same whatever be the quantity of the friction. If the body be the circumference of a circle, it will always lose half the velocity before its motion becomes uniform.

CASE II. 1. Let the body, besides having a progressive velocity in the direction LM (fig. 3.) have also a rotatory motion about its center in the direction gfe , and let v represent the initial velocity of any point of the circumference about the center, and suppose it first to be less than a ; then friction being a uniformly retarding force, no alteration of the velocity of the point of contact of the body upon the plane can affect the quantity of friction; hence the progressive velocity of the body will be the same as before, and consequently the rotatory velocity

city generated by friction will also be the same, to which if we add the velocity about the center at the beginning of the motion, we shall have the whole rotatory motion; hence therefore, $v + \frac{ra}{rs} \times a - \sqrt{a^2 - 2Fz} = \sqrt{a^2 - 2Fz}$, consequently $z = \frac{a^2 \times as^2 - v \times rs + a \times ra^2}{2F \times as^2}$ the space described before the motion becomes uniform.

2. If this value of z be substituted into the expression for the velocity, we shall have $\frac{v \times rs + a \times ra}{as}$ for the velocity when the friction ceases.

3. If $v = a$, then $z = 0$, and hence the body will continue to move uniformly with the first velocity.

4. If v be greater than a , then the rotatory motion of the point a on the plane being greater than its progressive motion and in a contrary direction, the absolute motion of the point a upon the plane will be in the direction ML , and consequently friction will now act in the direction LM in which the body moves, and therefore will accelerate the *progressive* and retard the *rotatory* motion; hence it appears, that *the progressive motion of a body may be ACCELERATED by friction*. Now to determine the space described before the motion becomes uniform, we may observe, that as the progressive motion of the body is now accelerated, the velocity after it has described any space z will be $= \sqrt{a^2 + 2Fz}$, hence the velocity acquired $= \sqrt{a^2 + 2Fz} - a$, and consequently the rotatory velocity destroyed $\frac{ra}{rs} \times \sqrt{a^2 + 2Fz} - a$, hence $v - \frac{ra}{rs} \times \sqrt{a^2 + 2Fz} - a = \sqrt{a^2 + 2Fz}$, therefore $z = \frac{rs \times v + ra \times a^2 - a^2 \times as^2}{2F \times as^2}$ the space required.

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5. If $a=0$, or the body be placed upon the plane without any progressive velocity, then $z = \frac{rs^2 \times v^2}{2F \times as^2}$.

CASE III. 1. Let the given rotatory motion be in the direction gef ; then as the friction must in this case always act in the direction ML , it must continually tend to destroy both the progressive and rotatory motion. Now as the velocity destroyed in the same time is in proportion to the retarding force, and the force which retards the *rotatory* is to the force which retards the *progressive* velocity by Cor. 2. Prop. 1. as $ra : rs$, therefore if v be to a as ra is to rs , then the retarding forces being in proportion to the velocities, both motions will be destroyed together, and consequently the body, after describing a certain space, will rest; which space, being that described by the body uniformly retarded by the force F , will, from what was proved in Case I. be equal to $\frac{a^2}{2F}$.

2. If v bears a greater proportion to a than ra does to rs , it is manifest, that the rotatory motion will not be all destroyed when the progressive is; consequently the body, after it has described the space $\frac{a^2}{2F}$, will return back in the direction ML ; for the progressive motion being then destroyed, and the rotatory motion still continuing in the direction gef , will cause the body to return with an accelerative velocity until the friction ceases by the body's beginning to roll, after which it will move on uniformly. Now to determine the space described before this happens, we have $rs : ra :: a : \frac{ra \times a}{rs}$ the rotatory velocity destroyed when the progressive is all lost; hence $v - \frac{ra \times a}{rs} = \frac{v \times rs - a \times ra}{rs}$ = the rotatory velocity at that time, which
being

being substituted for v in the last article of Case II. gives

$\frac{v+rs-a \times ra^2}{2F \times as^2}$ for the space described before the motion becomes uniform.

3. If v has a less proportion to a than ra has to rs , it is manifest, that the *rotatory* motion will be destroyed before the *progressive*; in which case a rotatory motion will be generated in a contrary direction until the two motions become equal, when the friction will instantly cease, and the body will then move on uniformly. Now $ra : rs :: v : \frac{v \times rs}{ra}$ the progressive velocity destroyed when the rotatory velocity ceases, hence $a - \frac{v \times rs}{ra} = \frac{a \times ra - v \times rs}{ra}$ = progressive velocity when it begins its rotatory motion in a contrary direction; substitute therefore this quantity for a in the expression for z in Case I. and we have

$\frac{rs^2 + 2rs \times ra \times a \times ra - v \times rs^2}{as^2 \times ar^2 \times 2F}$ for the space described after the rotatory motion ceases before the motion of the body becomes uniform. Now to determine the space described before the rotatory motion was all destroyed, we have (as the space from the end of a uniformly retarded motion varies as the square of

the velocity) $a^2 : \frac{a^2}{2F} :: \frac{a \times ra - v \times rs}{ra^2} : \frac{a \times ra - v \times rs}{2F \times ra^2}$ the space that could have been described from the time that the rotatory velocity was destroyed, until the progressive motion would have been destroyed had the friction continued to act; hence $\frac{a^2}{2F} - \frac{a \times ra - v \times rs}{2F \times ra^2} = \frac{2av \times ra \times rs - v^2 \times rs^2}{2F \times ra^2}$ = the space described when the rotatory motion was all destroyed, hence

$\frac{rs^2 + 2rs \times ra \times a \times ra - v \times rs^2}{as^2 \times ar^2 \times 2F} + \frac{2av \times ra \times rs - v^2 \times rs^2}{2F \times ra^2}$ = whole space described by the body before its motion becomes uniform.

DEFINITION.

The CENTER of FRICTION is that point in the base of a body on which it revolves, into which if the whole surface of the base, and the mass of the body were collected, and made to revolve about the center of the base of the given body, the angular velocity destroyed by its friction would be equal to the angular velocity destroyed in the given body by its friction in the same time.

PROPOSITION III.

To find the center of friction.

Let FGH (fig. 4.) be the base of a body revolving about its center C, and suppose about a, b, c , &c. to be indefinitely small parts of the base, and let A, B, C, &c. be the corresponding parts of the solid, or the prismatic parts having a, b, c , &c. for their bases; and P the center of friction. Now it is manifest, that the decrement of the angular velocity must vary as the whole diminution of the momentum of rotation caused by the friction *directly*, and as the whole momentum of rotation or effect of the inertia of all the particles of the solid *inversely*; the *former* being employed in diminishing the angular velocity, and the *latter* in opposing that diminution by the endeavour of the particles to persevere in their motion. Hence, if the effect of the friction varies as the effect of the inertia, the decrements of the angular velocity in a given time will be equal. Now as the quantity of friction (as has been proved from experiments) does not depend on the velocity, the effect of the friction of the elementary parts of the base a, b, c , &c.

will be as $a \times aC$, $b \times bC$, $c \times cC$, &c. also the effect of the inertia of the corresponding parts of the body will be as $A \times aC^2$, $B \times bC^2$, $C \times cC^2$, &c. Now when the whole surface of the base and mass of the body are concentrated in P, the effect of the friction will be as $\overline{a+b+c+\&c.} \times CP$, and of the inertia as $\overline{A+B+C+\&c.} \times CP^2$; consequently $a \times aC + b \times bC + c \times cC + \&c. : \overline{a+b+c+\&c.} \times CP :: A \times aC^2 + B \times bC^2 + C \times cC^2 + \&c. : \overline{A+B+C+\&c.} \times CP^2$; and hence

$$CP = \frac{A \times aC^2 + B \times bC^2 + C \times cC^2 + \&c. \times \overline{a+b+c+\&c.}}{a \times aC + b \times bC + c \times cC + \&c. \times \overline{A+B+C+\&c.}} = \text{(if } S = \text{the sum}$$

of the products of each particle into the square of its distance from the axis of motion, $T = \text{the sum of the products of each part of the base into its distance from the center, } s = \text{the area of the base, } t = \text{the solid content of the body)} \frac{S \times s}{T \times t}.$

PROPOSITION IV.

Given the velocity with which a body begins to revolve about the center of its base, to determine the number of revolutions which the body will make before all its motion be destroyed.

Let the friction, expressed by the velocity which it is able to destroy in the body if it were projected in a right line horizontally in one second, be determined by experiment, and called F; and suppose the initial velocity of the center of friction P about C to be a . Then conceiving the whole surface of the base and mass of the body to be collected into the point P, and (as has been proved in Prop. II.) $\frac{a^2}{2F}$ will be the space which the body so concentrated will describe before all its motion be destroyed;

hence if we put $z = PC$, $p =$ the circumference of a circle whose radius is unity, then will $pz =$ circumference described by the point P ; consequently $\frac{a^2}{2pzF} =$ the number of revolutions required.

Cor. If the solid be a cylinder and r be the radius of its base, then $z = \frac{3r}{4}$, and therefore the number of revolutions $= \frac{2a^2}{3prF}$.

PROPOSITION V.

To find the nature of the curve described by any point of a body affected by friction, when it descends down any inclined plane.

Let efg (fig. 5.) be the body, the points a, r, s , as in Prop. I. and conceive st, rn , to be two indefinitely small spaces described by the points s and r in the same time, and which therefore will represent the velocities of those points; but from Prop. I. the ratio of these velocities is expressed by $m \times CB : a \times CA$, hence $st : rn :: m \times CB : a \times CA$. With the center r let a circle vw be described touching the plane LM which is parallel to AC at the point b , and let the radius of this circle be such that, conceiving it to descend upon the plane LM along with the body descending on CA , the point b may be at rest, or the circle may roll without sliding. To determine which radius, produce rs to x , parallel to which draw ndy , and produce nt to z ; now it is manifest, that in order to answer the conditions above-mentioned, the velocity of the point x must be to the velocity of the point r as $2 : 1$, that is, $zx : yx :: 2 : 1$, hence $zy = yx = nr$. Now $xy : dt (:: ny : nd) :: rx : rs$; therefore $dt = \frac{rs}{rx} \times xy = \frac{rs}{rx} \times nr$, hence $ts (= td + ds = td + nr =$

Fig. 1.

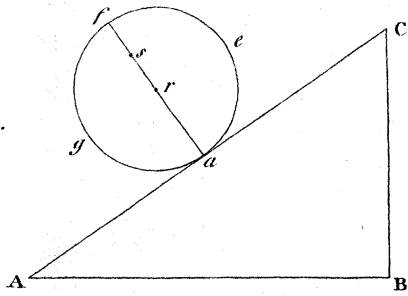


Fig. 2.

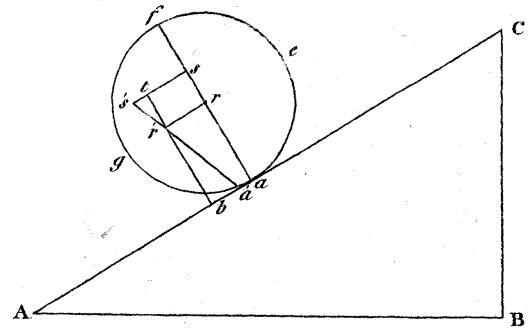


Fig. 3.

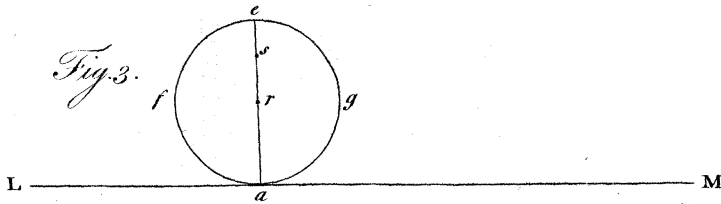


Fig. 5.

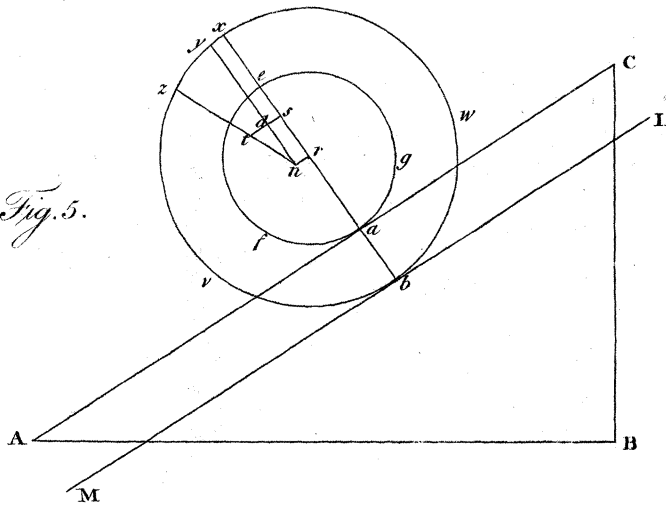
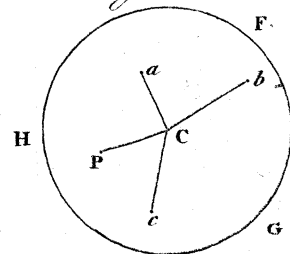


Fig. 4.



$\frac{rs}{rx} \times nr + nr) = \frac{rs+rx}{rx} \times nr$, consequently $\frac{rs+rx}{rx} : 1 :: ts : nr ::$

(from what is proved above) $m \times CB : a \times CA$; therefore $a \times CA \times rs + a \times CA \times rx = m \times CB \times rx$, hence $rx =$

$\frac{a \times CA \times rs}{m \times CB - a \times CA}$ the radius of the circle which rolling down

the inclined plane LM, and carrying the body with it, will give the true ratio of its progressive to its rotatory motion, and consequently that point of the circle which coincides with any given point of the body will, as the circle revolves upon the line LM, describe the same curve as the corresponding point of the body; but as the nature of the curve described by any point of a circle revolving upon a straight line is already very well known, it seems unnecessary to give the investigation.

By a method of reasoning, not very different, may the nature of the curve, which is described by any point of a body moving upon an horizontal plane, and affected by friction, be determined.

